Natural FLRW metrics on the Lie group of nonzero quaternions

Vladimir Trifonov
trifonov@member.ams.org

Abstract

It is shown that the Lie group of invertible elements of the quaternion algebra carries a family of natural closed Friedmann-Lemaître-Robertson-Walker metrics.

Introduction.

The quaternion algebra $\mathbb{H}$ is one of the most important and well-studies objects in mathematics (e. g. [Wid02] and references therein) and physics (e. g. [Adl95] and references therein). It has a natural Hermitian form which induces a Euclidean inner product on its additive vector space $S_{\mathbb{H}}$. There is also a family of natural Minkowski inner products (signature 2) on $S_{\mathbb{H}}$, induced by the structure tensor $\mathbf{H}$ of the quaternion algebra. This result was obtained in [Tri95], where a notion of a natural inner product on a linear algebra over a field $\mathbb{F}$ was introduced. The result came out of a study of relationship between natural metric properties of unital algebras and internal logic of topoi they generate. It was shown in [Tri95] that if the logic of a topos is bivalent Boolean then the generating algebra is isomorphic to the quaternion algebra with a family of Minkowski inner products. In this note we show that for a unital algebra the inner products can be naturally extended over the Lie group of its invertible elements, producing a family of *principal metrics*. In particular, for the quaternion algebra, these metrics are closed Friedmann-Lemaître-Robertson-Walker. These metrics are of interest because they constitute one of the most important classes (as far as *our* universe is concerned) of solutions of Einstein's equations, and there are

indications in astrophysics and cosmology that the universe may be spatially closed ([Tr03] and references therein).

Remark 0.1. Some of the notations are slightly nonstandard. Small Greek indices, α, β, γ and small Latin indices p, q *always* run 0 to 3 and 1 to 3, respectively. Summation is assumed on repeated indices of different levels. We use the $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]$ device to denote tensor ranks; for example a one-form is a $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$-tensor. For clarity of the exposition we use $\Box$ at the end of a *Proof*, and each *Remark* ends with the sign appearing at the end of this line. $\diamond$

Definition 0.1. An $\mathbb{F}$-*algebra*, $\mathbb{A}$, is an ordered pair $(S_{\mathbb{A}}, \mathbf{A})$, where $S_{\mathbb{A}}$ is a vector space over a field $\mathbb{F}$, and $\mathbf{A}$ is a $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$-tensor on $S_{\mathbb{A}}$, called the *structure tensor* of $\mathbb{A}$. Each vector $\boldsymbol{a}$ of $S_{\mathbb{A}}$ is called an *element* of $\mathbb{A}$, denoted $a \in \mathbb{A}$. The *dimensionality* of $\mathbb{A}$ is that of $S_{\mathbb{A}}$.

Remark 0.2. This is an unconventional definition of a linear algebra over $\mathbb{F}$. Indeed, the tensor $\mathbf{A}$ induces a binary operation $S_{\mathbb{A}} \times S_{\mathbb{A}} \to S_{\mathbb{A}}$, called the *multiplication* of $\mathbb{A}$: to each pair of vectors $(\boldsymbol{a}, \boldsymbol{b})$ the tensor $\mathbf{A}$ associates a vector $\boldsymbol{ab} : S_{\mathbb{A}}^* \to \mathbb{F}$, such that $(\boldsymbol{ab})(\tilde{\boldsymbol{\tau}}) = \mathbf{A}(\tilde{\boldsymbol{\tau}}, \boldsymbol{a}, \boldsymbol{b}), \forall \tilde{\boldsymbol{\tau}} \in S_{\mathbb{A}}^*$. An $\mathbb{F}$-algebra with an associative multiplication is called *associative*. An element $\boldsymbol{\imath}$, such that $\boldsymbol{a\imath} = \boldsymbol{\imath a} = \boldsymbol{a}, \forall \boldsymbol{a} \in \mathbb{A}$ is called an *identity* of $\mathbb{A}$. $\diamond$

Definition 0.2. For an $\mathbb{F}$-algebra $\mathbb{A}$ and a nonzero one-form $\tilde{\boldsymbol{\tau}} \in S_{\mathbb{A}}^*$, a *principal inner product* is a $\left[\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right]$-tensor, $\mathbf{A}\,[\tilde{\boldsymbol{\tau}}]$, on $S_{\mathbb{A}}$, assigning to each ordered pair $(\boldsymbol{a}, \boldsymbol{b})$ a number $\mathbf{A}\,[\tilde{\boldsymbol{\tau}}](\boldsymbol{a}, \boldsymbol{b}) := \mathbf{A}(\tilde{\boldsymbol{\tau}}, \boldsymbol{a}, \boldsymbol{b}) \in \mathbb{F}$, just in case it is symmetric, $\mathbf{A}\,[\tilde{\boldsymbol{\tau}}](\boldsymbol{a}, \boldsymbol{b}) = \mathbf{A}\,[\tilde{\boldsymbol{\tau}}](\boldsymbol{b}, \boldsymbol{a}), \forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{A}$.

Remark 0.3. In other words, a principal inner product is the contraction of a one-form with the structure tensor. $\diamond$

Definition 0.3. For each $\mathbb{F}$-algebra $\mathbb{A} = (S_{\mathbb{A}}, \mathbf{A})$, an $\mathbb{F}$-algebra $[\mathbb{A}] = (S_{\mathbb{A}}, [\mathbf{A}])$, with the structure tensor defined by

$$[\mathbf{A}]\,(\tilde{\boldsymbol{\tau}}, \boldsymbol{a}, \boldsymbol{b}) := \mathbf{A}(\tilde{\boldsymbol{\tau}}, \boldsymbol{a}, \boldsymbol{b}) - \mathbf{A}(\tilde{\boldsymbol{\tau}}, \boldsymbol{b}, \boldsymbol{a}), \forall \tilde{\boldsymbol{\tau}} \in S_{\mathbb{A}}^*, \boldsymbol{a}, \boldsymbol{b} \in \mathbb{A},$$

is called the *commutator* algebra of $\mathbb{A}$.

Definition 0.4. A finite dimensional associative $\mathbb{R}$-algebra with an identity is called a *unital* algebra.

Lemma 0.1. *The set $\mathcal{A}$ of all invertible elements of a unital algebra $\mathbb{A}$ is a Lie group with respect to the multiplication of $\mathbb{A}$, with $[\mathbb{A}]$ as its Lie algebra.*

Proof. See, for example, [Pos82] for a proof of this simple lemma. □

Remark 0.4. For an $\mathbb{R}$-algebra $\mathbb{A}$, its vector space $S_{\mathbb{A}}$ canonically generates a (linear) manifold $\mathcal{S}_{\mathbb{A}}$ with the same carrier, with a bijection $\mathcal{J} : \mathcal{S}_{\mathbb{A}} \to S_{\mathbb{A}}$. We use the normal $(a, u, ...)$ and bold $(\boldsymbol{a}, \boldsymbol{u}...)$ fonts, respectively, to denote their elements, e. g., $\mathcal{J}(a) = \boldsymbol{a}$. The tangent space $T_a\mathcal{S}_{\mathbb{A}}$ is identified with $S_{\mathbb{A}}$ at each point $a \in \mathcal{S}_{\mathbb{A}}$ via an isomorphism $\mathcal{J}^*_a : T_a\mathcal{S}_{\mathbb{A}} \to S_{\mathbb{A}}$ sending a tangent vector to the curve $\mu : \mathbb{R} \to \mathcal{S}_{\mathbb{A}}, \mu(t) = a + tu$, at the point $\mu(0) = a \in \mathcal{S}_{\mathbb{A}}$, to the vector $\boldsymbol{u} \in S_{\mathbb{A}}$, with the "total" map $\mathcal{J}^* : T\mathcal{S}_{\mathbb{A}} \to S_{\mathbb{A}}$. A linear map $\boldsymbol{F} : S_{\mathbb{A}} \to S_{\mathbb{A}}$ induces a vector field $\boldsymbol{f} : \mathcal{S}_{\mathbb{A}} \to T\mathcal{S}_{\mathbb{A}}$ on $\mathcal{S}_{\mathbb{A}}$, such that the following diagram commutes,

$$
\begin{array}{ccc}
\mathcal{S}_{\mathbb{A}} & \xrightarrow{\boldsymbol{f}} & T\mathcal{S}_{\mathbb{A}} \\
{\scriptstyle \mathcal{J}}\downarrow & & \downarrow{\scriptstyle \mathcal{J}^*} \\
S_{\mathbb{A}} & \xrightarrow{\boldsymbol{F}} & S_{\mathbb{A}}
\end{array} \quad . \tag{1}
$$

For a unital algebra $\mathbb{A}$ the Lie group $\mathcal{A}$ is a submanifold of $\mathcal{S}_{\mathbb{A}}$, with the inclusion map $\bar{\mathcal{J}} : \mathcal{A} \to \mathcal{S}_{\mathbb{A}}$, which is the restiction, to $\mathcal{A}$, of the identity map. ◇

Remark 0.5. For each basis $(\boldsymbol{e}_j)$ on the vector space $S_{\mathbb{A}}$ of a unital algebra, there is a natural basis field on $\mathcal{A}$, namely the basis $(\hat{\boldsymbol{e}}_j)$ of left invariant vector fields generated by $(\boldsymbol{e}_j)$. We call $(\hat{\boldsymbol{e}}_j)$ a *proper frame generated by* $(\boldsymbol{e}_j)$. The value, $(\hat{\boldsymbol{e}}_j)(a)$, of $(\hat{\boldsymbol{e}}_j)$ at a is basis on the tangent space $T_a\mathcal{A}$; it is referred to as a *proper basis* (at a) generated by $(\boldsymbol{e}_j)$. In particular, $(\hat{\boldsymbol{e}}_j)(\imath)$, the proper basis at the identity generated by $(\boldsymbol{e}_j)$ coincides with $(\boldsymbol{e}_j)$. ◇

Definition 0.5. For a unital algebra $\mathbb{A}$, let $(\hat{\boldsymbol{e}}_j)$ be a proper frame on $\mathcal{A}$, generated by a basis $(\boldsymbol{e}_j)$ on $S_{\mathbb{A}}$. The *structure field* of the Lie group $\mathcal{A}$ is a tensor field $\boldsymbol{\mathcal{A}}$ on $\mathcal{A}$, assigning to each point $a \in \mathcal{A}$ a $[\frac{1}{2}]$-tensor $\boldsymbol{\mathcal{A}}(a)$ on $T_a\mathcal{A}$, with components $\mathcal{A}^i_{jk}(a) := (\mathcal{A}(a))^i_{jk}$ in the basis $(\hat{\boldsymbol{e}}_j)(a)$, defined by

$$
\mathcal{A}^i_{jk}(a) := \mathsf{A}^i_{jk}, \quad \forall a \in \mathcal{A},
$$

where A^i_{jk} are the components of the structure tensor $\mathbf{A}$ in the basis $(\boldsymbol{e}_j)$.

Remark 0.6. Intuitively, the structure field is the constant extension of the structure tensor along the left invariant vector fields. ◇

Definition 0.6. For a unital algebra $\mathbb{A}$ and each $a \in \mathcal{A}$, an $\mathbb{R}$-algebra $\mathbb{A}\{a\} = (S_{\mathbb{A}\{a\}}, \mathsf{\mathbf{A\{}}a\mathsf{\mathbf{\}}})$, where $S_{\mathbb{A}\{a\}} := T_a\mathcal{A}$, and $\mathsf{\mathbf{A\{}}a\mathsf{\mathbf{\}}} := \boldsymbol{\mathcal{A}}(a)$, is called the *tangent algebra* of the Lie group $\mathcal{A}$ at a.

Remark 0.7. It is easy to see that for each $a \in \mathcal{A}$, the tangent algebra $\mathbb{A}\{a\}$ is isomorphic to $\mathbb{A}$; in particular, each $\mathbb{A}\{a\}$ is unital. $\diamond$

Definition 0.7. For a unital algebra $\mathbb{A}$ and a twice differentiable real function $\mathfrak{T}$ on the Lie group $\mathcal{A}$, a *principal metric on* $\mathcal{A}$ is a $\left[{0 \atop 2}\right]$-tensor field $\boldsymbol{\mathfrak{T}}$ on $\mathcal{A}$, such that that $\boldsymbol{\mathfrak{T}}(a) = \mathsf{\mathbf{A\{}}a\mathsf{\mathbf{\}}}\,[\tilde{\boldsymbol{a}}], \forall a \in \mathcal{A}$, where $\tilde{\boldsymbol{a}} := d\mathfrak{T}(a)$ is the value of the gradient of $\mathfrak{T}$ at a.

Remark 0.8. In other words, a principal metric is the contraction of a one-form field on $\mathcal{A}$ with the structure field of $\mathcal{A}$. For each $a \in \mathcal{A}$, the value, $\boldsymbol{\mathfrak{T}}(a)$, of $\boldsymbol{\mathfrak{T}}$ is a principal inner product on the tangent algebra $\mathbb{A}\{a\}$. $\diamond$

1 Quaternion algebra.

Definition 1.1. A four dimensional $\mathbb{R}$-algebra, $\mathbb{H} = (S_{\mathbb{H}}, \mathsf{\mathbf{H}})$, is called a *quaternion algebra* (with *quaternions* as its elements), if there is a basis on $S_{\mathbb{H}}$, in which the components of the structure tensor $\mathsf{\mathbf{H}}$ are given by the entries of the following matrices,

$$\mathsf{H}^0_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ \mathsf{H}^1_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$
$$\mathsf{H}^2_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ \mathsf{H}^3_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (2)$$

We refer to such a basis as *canonical.*

Remark 1.1. The vectors of the canonical basis are denoted $\mathbf{1}$, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. A quaternion algebra is unital, with the first vector of the canonical basis, $\mathbf{1}$, as its identity. Since $(\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ is a basis on a real vector space, any quaternion a can be presented as $a^0\mathbf{1} + a^1\boldsymbol{i} + a^2\boldsymbol{j} + a^3\boldsymbol{k}, a^\beta \in \mathbb{R}$. A quaternion $\bar{a} = a^0\mathbf{1} - a^1\boldsymbol{i} - a^2\boldsymbol{j} - a^3\boldsymbol{k}$ is called *conjugate* to a. We refer to a^0 and $a^p\boldsymbol{i}_p$ as

the *real* and *imaginary part* of a, respectively. Quaternions of the form $a^0\mathbf{1}$ are in one-to-one correspondence with real numbers, which is often denoted, with certain notational abuse, as $\mathbb{R} \subset \mathbb{H}$. $\diamond$

Remark 1.2. A linear transformation $S_\mathbb{H} \to S_\mathbb{H}$ with the following components in the canonical basis,

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathsf{B} \end{pmatrix}, \mathsf{B} \in SO(3),$$

takes $(\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ to a basis $(\boldsymbol{i}_\beta)$ in which the components (2) of the structure tensor will *not* change, and neither will the multiplicative behavior of vectors of $(\boldsymbol{i}_\beta)$. Thus, we have a class of canonical bases parameterized by elements of $SO(3)$. $\diamond$

Lemma 1.1. *Every principal inner product on $\mathbb{H}$ is Minkowski.*

Proof. For the quaternion algebra the components of the structure tensor H in a canonical basis are given by (2). A one-form $\tilde{\boldsymbol{\tau}}$ on $S_\mathbb{H}$ with components $\tilde{\tau}_\beta$ in (the dual of) a canonical basis $(\boldsymbol{i}_\beta)$ contracts with the structure tensor into a $\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$-tensor on $S_\mathbb{H}$ with the following components in the basis $(\boldsymbol{i}_\beta)$:

$$\begin{pmatrix} \tilde{\tau}_0 & \tilde{\tau}_1 & \tilde{\tau}_2 & \tilde{\tau}_3 \\ \tilde{\tau}_1 & -\tilde{\tau}_0 & \tilde{\tau}_3 & -\tilde{\tau}_2 \\ \tilde{\tau}_2 & -\tilde{\tau}_3 & -\tilde{\tau}_0 & \tilde{\tau}_1 \\ \tilde{\tau}_3 & \tilde{\tau}_2 & -\tilde{\tau}_1 & -\tilde{\tau}_0 \end{pmatrix}.$$

The only way to make this symmetric is to put $\tilde{\tau}_1 = -\tilde{\tau}_1$, $\tilde{\tau}_2 = -\tilde{\tau}_2$, $\tilde{\tau}_3 = -\tilde{\tau}_3$, which yields $\tilde{\tau}_1 = \tilde{\tau}_2 = \tilde{\tau}_3 = 0$:

$$(\mathsf{H}\,[\tilde{\boldsymbol{\tau}}])_{\alpha\beta} = \begin{pmatrix} \tilde{\tau}_0 & 0 & 0 & 0 \\ 0 & -\tilde{\tau}_0 & 0 & 0 \\ 0 & 0 & -\tilde{\tau}_0 & 0 \\ 0 & 0 & 0 & -\tilde{\tau}_0 \end{pmatrix}. \tag{3}$$

. $\square$

2 Natural structures on $\mathcal{H}$.

There is a class of canonical bases on $S_\mathbb{H}$ (see Remark 1.2) whose members differ from one another by a rotation in the hyperplane of pure imaginary

quaternions. Each canonical basis $(\boldsymbol{i}_\beta)$ on $S_{\mathbb{H}}$ induces a *canonical* coordinate system $(w,\, x,\, y,\, z)$ on the linear manifold $\mathcal{S}_{\mathbb{H}}$ canonically generated by $S_{\mathbb{H}}$, and therefore also on its submanifold $\mathcal{H}$ of nonzero quaternions: a point $a \in \mathcal{H}$ such that $\mathcal{J}(a) = \boldsymbol{a} = a^\beta \boldsymbol{i}_\beta$ is assigned coordinates $(w = a^0,\, x = a^1,\, y = a^2,\, z = a^3)$. This coordinate system covers both $\mathcal{S}_{\mathbb{H}}$ and $\mathcal{H}$ with a single patch. Since $\mathbf{0} \notin \mathcal{H}$, at least one of the coordinates is always nonzero for any point $a \in \mathcal{H}$. For a differentiable function $R : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ there is a system of natural spherical coordinates $(\eta,\, \chi,\, \theta,\, \varphi)$ on $\mathcal{H}$, related to the canonical coordinates by

$$
\begin{aligned}
w = R(\eta)\cos\chi, \quad x = R(\eta)\sin\chi\sin\theta\cos\varphi,& \\
y = R(\eta)\sin\chi\sin\theta\sin\varphi, \quad z = R(\eta)\sin\chi\cos\theta.&
\end{aligned}
$$

Each canonical basis $(\boldsymbol{i}_\beta)$ can be considered a basis on the vector space of the Lie algebra of $\mathcal{H}$, i. e., the tangent space $T_{\mathbf{1}}\mathcal{H} \cong S_{\mathbb{H}}$ to $\mathcal{H}$ at the point (1, 0, 0, 0), the identity of the group $\mathcal{H}$. There are several natural basis fields on $\mathcal{H}$ induced by each basis $(\boldsymbol{i}_\beta)$. First of all, we have the proper frame $(\hat{\boldsymbol{\imath}}_\beta)$, of left invariant vector fields on $\mathcal{H}$ (see Remark 0.5), which is a *noncoordinate* basis field. There are also two *coordinate* basis fields, the *canonical frame*, $(\partial_w,\, \partial_x,\, \partial_y,\, \partial_z)$ and the corresponding *spherical frame* $(\partial^R_\eta,\, \partial^R_\chi,\, \partial^R_\theta,\, \partial^R_\varphi)$. A left invariant vector field $\hat{\boldsymbol{u}}$ on $\mathcal{H}$, generated by a vector $\boldsymbol{u} \in S_{\mathbb{H}}$ with components (u^β) in a canonical basis, associates to each point $a \in \mathcal{H}$ with coordinates $(w,\, x,\, y,\, z)$ a vector $\hat{\boldsymbol{u}}(a) \in T_a\mathcal{H}$ with the components $\hat{u}^\beta(a) = (\boldsymbol{a}\boldsymbol{u})^\beta$ in the basis $(\partial_w,\, \partial_x,\, \partial_y,\, \partial_z)(a)$ on $T_a\mathcal{H}$:

$$
\begin{aligned}
\hat{u}^0(a) = wu^0 - xu^1 - yu^2 - zu^3, \quad \hat{u}^1(a) = wu^1 + xu^0 + yu^3 - zu^2,& \\
\hat{u}^2(a) = wu^2 - xu^3 + yu^0 + zu^1, \quad \hat{u}^3(a) = wu^3 + xu^2 - yu^1 + zu^0.& \qquad (4)
\end{aligned}
$$

The system (4) contains sufficient information to compute transformation between the frames. For example, the transformation between the spherical and proper frames is given by

$$
\begin{pmatrix}
R/\dot{R} & 0 & 0 & 0 \\
0 & \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\
0 & \frac{\cos\chi\cos\theta\cos\varphi + \sin\chi\sin\varphi}{\sin\chi} & \frac{\cos\chi\cos\theta\sin\varphi + \sin\chi\cos\varphi}{\sin\chi} & \frac{\cos\chi\sin\theta}{\sin\chi} \\
0 & \frac{\sin\chi\cos\theta\cos\varphi - \cos\chi\sin\varphi}{\sin\chi\sin\theta} & \frac{\sin\chi\cos\theta\sin\varphi + \cos\chi\cos\varphi}{\sin\chi\sin\theta} & -1
\end{pmatrix},
$$

where $\dot{R} := \frac{dR}{d\eta} : \mathbb{R} \to \mathbb{R} \setminus \{0\}$.

Definition 2.1. A Lorentzian metric on a four dimensional manifold is called *closed FLRW* (Friedmann-Lemaître-Robertson-Walker) if there is a coordinate system (x^β), such that in the corresponding coordinate frame the components of the metric are given by the entries of the following matrix:

$$\pm\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathsf{a}^2 & 0 & 0 \\ 0 & 0 & -\mathsf{a}^2\sin^2(x^1) & 0 \\ 0 & 0 & 0 & -\mathsf{a}^2\sin^2(x^1)\sin^2(x^2) \end{pmatrix},$$

where $\mathsf{a} : \mathbb{R} \to \mathbb{R}$, referred to as the *scale factor*, is a function of x^0 only.

3 Principal metrics on $\mathcal{H}$.

Theorem 3.1. *Every principal metric on $\mathcal{H}$ is closed FLRW.*

Proof. Let $\tilde{\boldsymbol{\tau}}$ and $(\boldsymbol{\imath}_\beta)$ be a one-form and a canonical basis on $S_{\mathbb{H}}$, respectively. For each point $a \in \mathcal{H}$ the $\mathbb{R}$-algebra $\mathbb{H}(a)$ is the tangent algebra, at a, of the Lie group $\mathcal{H}$. For each $a \in \mathcal{H}$ the components of the structure tensor $\mathbf{H}\{a\}$ and a principal inner product, $\mathbf{H}\{a\}\,[\tilde{\boldsymbol{\tau}}]$, of $\mathbb{H}(a)$ in the basis $(\hat{\boldsymbol{\imath}}_\beta)(a)$ are given by (2) and (3), respectively. Therefore, the components of a principal metric, $\boldsymbol{\mathcal{T}}$, in the proper frame $(\hat{\boldsymbol{\imath}}_\beta)$ must have the form

$$\begin{pmatrix} \tilde{\tau} & 0 & 0 & 0 \\ 0 & -\tilde{\tau} & 0 & 0 \\ 0 & 0 & -\tilde{\tau} & 0 \\ 0 & 0 & 0 & -\tilde{\tau} \end{pmatrix}, \tag{5}$$

for some function $\tilde{\tau} : \mathcal{H} \to \mathbb{R} \setminus \{0\}$. In other words, any principal metric on $\mathcal{H}$ is obtained by contraction of a one-form field $\tilde{\boldsymbol{\tau}}$, whose components in $(\hat{\boldsymbol{\imath}}_\beta)$ are $(\tilde{\tau}, 0, 0, 0)$, with the structure field $\boldsymbol{\mathcal{H}}$. This one-form is exact, i. e., there exists a twice differentiable function $\mathcal{T}$, such that $d\mathcal{T} = \tilde{\boldsymbol{\tau}}$. In the spherical frame with $R(\eta) = \exp(\eta)$ the components of $\tilde{\boldsymbol{\tau}}$ are also $(\tilde{\tau},\ 0,\ 0,\ 0)$, and,

$$d\mathcal{T}_0 = \frac{\partial \mathcal{T}}{\partial \eta} = \tilde{\tau}, \quad d\mathcal{T}_1 = \frac{\partial \mathcal{T}}{\partial \chi} = d\mathcal{T}_2 = \frac{\partial \mathcal{T}}{\partial \theta} = d\mathcal{T}_3 = \frac{\partial \mathcal{T}}{\partial \varphi} = 0. \tag{6}$$

It follows from (6) that both $\mathcal{T}$ and $\tilde{\tau}$ depend on η only. Since $\frac{\partial \mathcal{T}}{\partial \eta}$ is differentiable, $\tilde{\tau}$ must be at least continuous. Since $\tilde{\tau}(\eta) \neq 0, \forall \eta \in \mathbb{R}$, $\tilde{\tau}$ cannot

change sign. Computing the components of the principal metric $\mathfrak{T}$ in the spherical frame we get

$$\mathcal{T}_{\alpha\beta} = \begin{pmatrix} \tilde{\tau}(\eta)(\frac{\dot{R}}{R})^2 & 0 & 0 & 0 \\ 0 & -\tilde{\tau}(\eta) & 0 & 0 \\ 0 & 0 & -\tilde{\tau}(\eta)\sin^2\chi & 0 \\ 0 & 0 & 0 & -\tilde{\tau}(\eta)\sin^2\chi\sin^2\theta \end{pmatrix}.$$

If $\tilde{\tau}(\eta) > 0$, we take $R(\eta)$ such that $\tilde{\tau}(\eta)(\frac{\dot{R}}{R})^2 = 1$, which yields

$$R(\eta) = exp\int \frac{d\eta}{\pm\sqrt{\tilde{\tau}(\eta)}} \quad . \tag{7}$$

In other words, with $R(\eta)$ satisfying (7), the components of the principal metric in the spherical frame are

$$\mathcal{T}_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathsf{a}^2 & 0 & 0 \\ 0 & 0 & -\mathsf{a}^2\sin^2\chi & 0 \\ 0 & 0 & 0 & -\mathsf{a}^2\sin^2\chi\sin^2\theta \end{pmatrix},$$

where the scale factor $\mathsf{a}(\eta) := \sqrt{\tilde{\tau}(\eta)}$.

If $\tilde{\tau}(\eta) < 0$, similar considerations show that the metric is also closed FLRW with the scale factor $\mathsf{a}(\eta) := \sqrt{-\tilde{\tau}(\eta)}$. □

Corollary 3.1. *$\mathcal{T}$ is a monotonous function of η for each principal metric $\mathfrak{T}$ of $\mathcal{H}$.*

Thus the natural geometry of the Lie group of nonzero quaternions $\mathcal{H}$ is defined by a family of closed Friedmann-Lemaître-Robertson-Walker metrics.

References

[Adl95] S. L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields*, Oxford University Press, Oxford, UK (1995).

[Pos82] М. М. Постников, Группы и алгебры Ли, Наука, Москва (1982).

[Tr03] R. Triay, *IJTP* **42** (6) (2003) 1187-1192.

[Tri95] V. Trifonov, *Europhys. Lett.* **32** (8), 1995, 621-626.

[Wid02] D. Widdows, *Quaternionic Algebraic Geometry*, PhD Thesis, St Annes's College, Oxford, UK (2002).